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Suppression of Chaos in a One-dimensional Mapping

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Abstract. The suppression of chaos in an one-dimensional model of malignant tumor growth is presented. As a result, a steady-state and different periodic motions, embedded in the chaotic attractor, are stabilized.

Key words: Controlling chaos, Deterministic chaos, Nonlinear dynamics, One-dimensional mapping, Malignant tumor.

1. Introduction

Chaotic systems are characterized by extreme sensitivity to the initial conditions. This characteristic has been considered as a troublesome property and for many years it was generally believed that chaotic motions are not controllable.

However, recent studies have demonstrated that chaotic systems permit the use of small perturbations to control trajectories. Now the ability to control chaotic motions is studied, both theoretically and experimentally, by many.

An efficient method for achieving control was proposed by Ott, Grebogi and York (OGY) [1]. By making small time-dependent perturbations in some accessible system parameters, they have shown that it is possible to obtain a regular or periodic motion. Thus, unstable periodic orbits embedded in a chaotic attractor can be stabilized. The OGY method and its variants have been applied to various experimental systems [2–4].

Another method of controlling chaos has been recently proposed by Guemez and Matias (GM) [5–10]. This method allows the stabilization of chaotic systems by applying proportional feedbacks to the system variables. It was applied in the case of a chemical system, but the authors consider possible applications of their method to biological systems.

Thus, the capability of controlling chaotic systems, without a parallel in non-chaotic systems, seems to be useful in the case of systems interacting with the environment, as in the case of life forms.

In the present study we apply the recently introduced chaos suppression method (GM) to a model which describes growth of malignant tumors.

Our attention was attracted to such a model for two reasons: first, the model, which is a simple one-dimensional mapping, but with a complex dynamics, can be easily integrated by computer and secondly, the control of chaos in the present instance, a theoretical problem, could offer a new way of understanding results of controlled interactions in biological systems.

2. The One-dimensional Model

There is observational evidence that malignant tumor growth can be modelled by chaotic dynamical systems (see for example [11]). The standard model is the Gompertz equation described by Swan [12]:

$$\frac{1}{N}\frac{\mathrm{d}N}{\mathrm{d}t} = a \ln\left(\frac{N_0}{N}\right), \tag{1}$$

where $N(t)$ is the number of cells of the tumor. This model fits well with many human tumors in observed regions, i.e. when $N(t) \geqslant 10^9$ cells. Recently, another model, biologically motivated, has been proposed by Ahmed [13]:

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -aN(t) + b[N(t)]^{2/3}, \tag{2}$$

which, with a suitable choice of the parameters a and b, describes the same tumor growth as Equation (1).

In this model the tumor is assumed to consist of a core and a surface, the cell growth is proportional to the surface and the cell loss to the volume. In deriving Equation (2) it was assumed that the surface area is proportional to the volume to the power 2/3. In fact this power can be considered as a function of the fractal dimension, if the tumor is approximated as a fractal.

The discrete version of Equation (2) is:

$$N_{t+1} = (1-a)N_t + bN_t^{2/3},$$

which, after rescaling, can be written as

$$N_{t+1} = c(N_t^{2/3} - N_t).$$

A more convenient form of this equation, for numerical investigation is the following equivalent one-dimensional mapping:

$$x_{n+1} = 6.75r(x_n^{2/3} - x_n) = f(r, x_n), \tag{3}$$

where the control parameter $r \in [0,1]$ and $f\colon [0,1] \to [0,1]$ for any $x_0 \in [0,1]$.

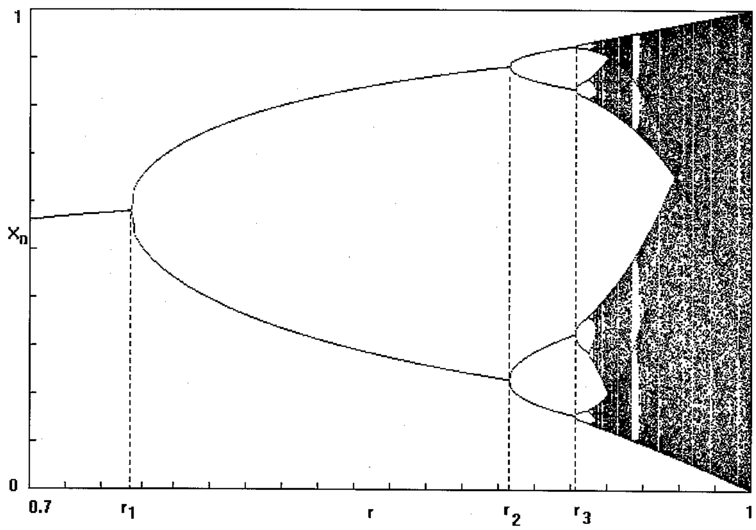

Figure 1. Bifurcation diagram of the map.

The fixed points of the mapping, which are the steady states of the system, must satisfy the equation $x_{n+1} = x_n$. Thus one obtains $x_1^* = 0$, a fixed point regardless of the value of r, and

$$x_2^* = \left(\frac{6.75r}{1 + 6.75r}\right)^3, \tag{4}$$

the second fixed point.

The nonzero steady state is stable if

$$|\lambda| = \left|\frac{\mathrm{d}f}{\mathrm{d}x}\right|_{x=x_2^*} < 1, \tag{5}$$

and this implies $0.148 < r < 0.740$. By numerical calculation one finds that $r = 0.740 = r_1$ is a bifurcation point and for $r > r_1$ the mapping has a 2-cycle, which is stable when $0.740 < r < 0.898$. At $r = 0.898 = r_2$ there is a new bifurcation for a 4-cycle. From the bifurcation diagram (Figure 1) another pitchfork bifurcation can be seen for an 8-cycle (at $r = 0.924 = r_3$) or for a 16-cycle.

From Figure 1 it is obvious that by increasing the value of the control parameter the system becomes chaotic after some value. The control of chaos implies taking into consideration the values of this parameter in this chaotic domain. We can also observe that the region of chaotic behavior is interrupted by intervals of periodic behavior (periodic windows). One of the largest windows where a periodic orbit (3-cycle) occurs is shown in Figure 2.

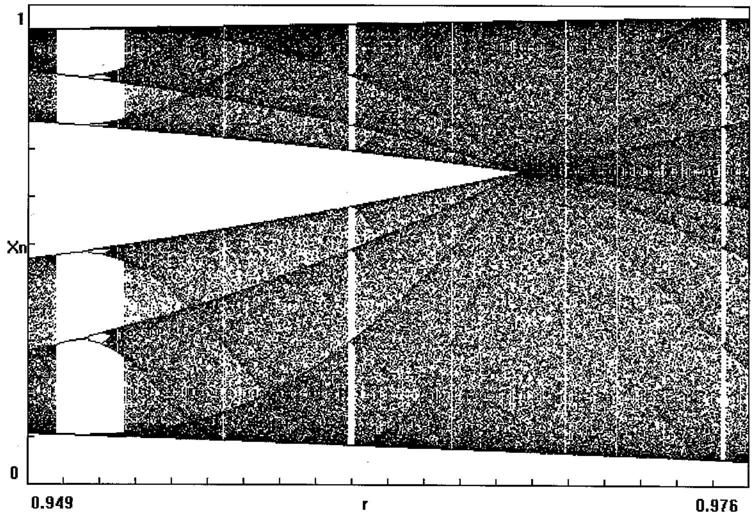

Figure 2. Magnification of the bifurcation diagram in the region of the 3-period window.

3. The Control Algorithm

The (GM) control algorithm consists in the application every Δn iterations of feedback to the variable x, of the form:

$$x_n \to x_n(1+\gamma), \tag{6}$$

where γ represents the strength of the feedback.

By this method, depending on the sign of γ, some part of x is injected or retired from (3), which depends on the value of x at that moment. In fact a new dynamical system is created based on the original one.

By this method, by appropriately choosing Δn and γ it is possible to stabilize different unstable periodic orbits.

4. Numerical Results

We have carried out many numerical investigations with different values of Δn and γ. In Figures 3–6 (a and b) we present some results where one-, two- and four-cycles have been stabilized. The values of $\Delta n, \gamma$ and of the control parameter, which has been chosen in the chaotic domain, are indicated. In every figure, part (a) shows a plot of x_n versus number of iterations, while in part (b) we have plotted x_{n+1} versus x_n. The first vertical dashed line splits part (a) of the Figures into two regions – before and after the action of the control algorithm. In part (b) of the Figures the known cobweb technique is illustrated. Thus, before the action of the control algorithm, one can see chaotic behavior

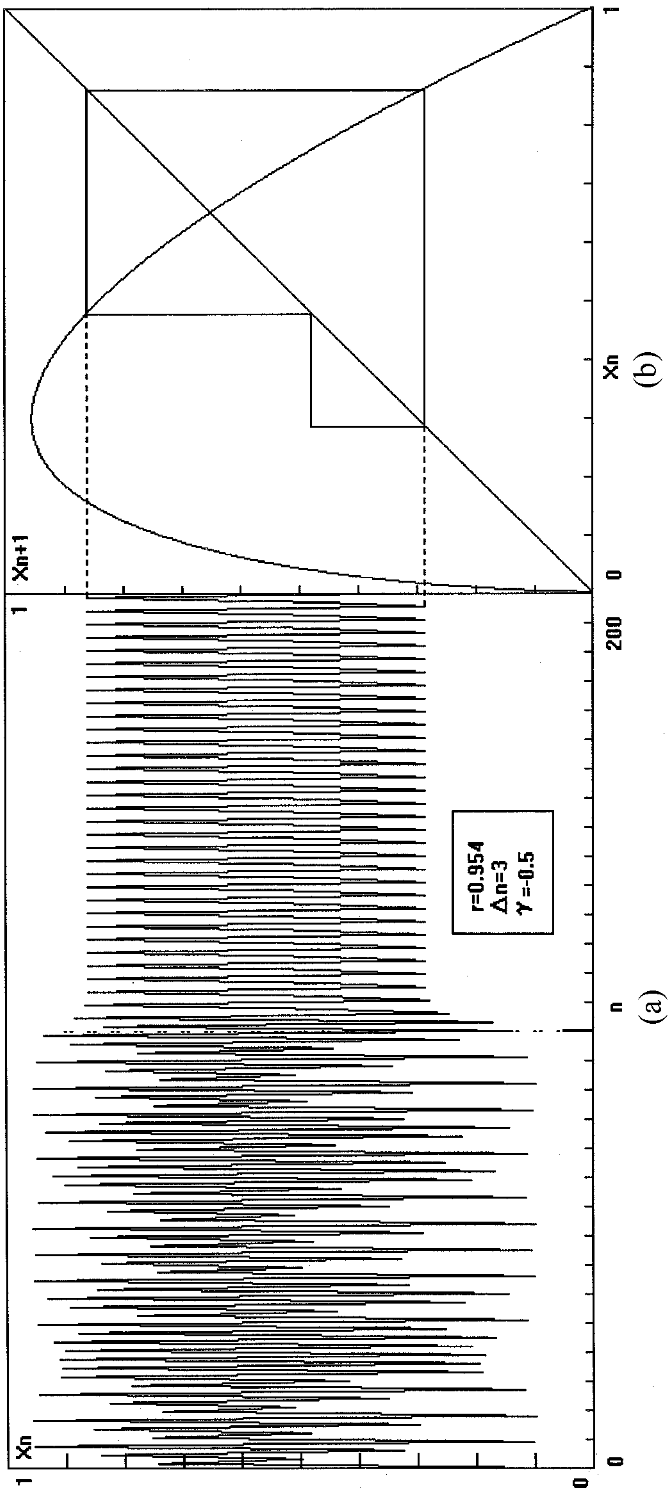

Figure 3. A stabilized one-period oscillation.

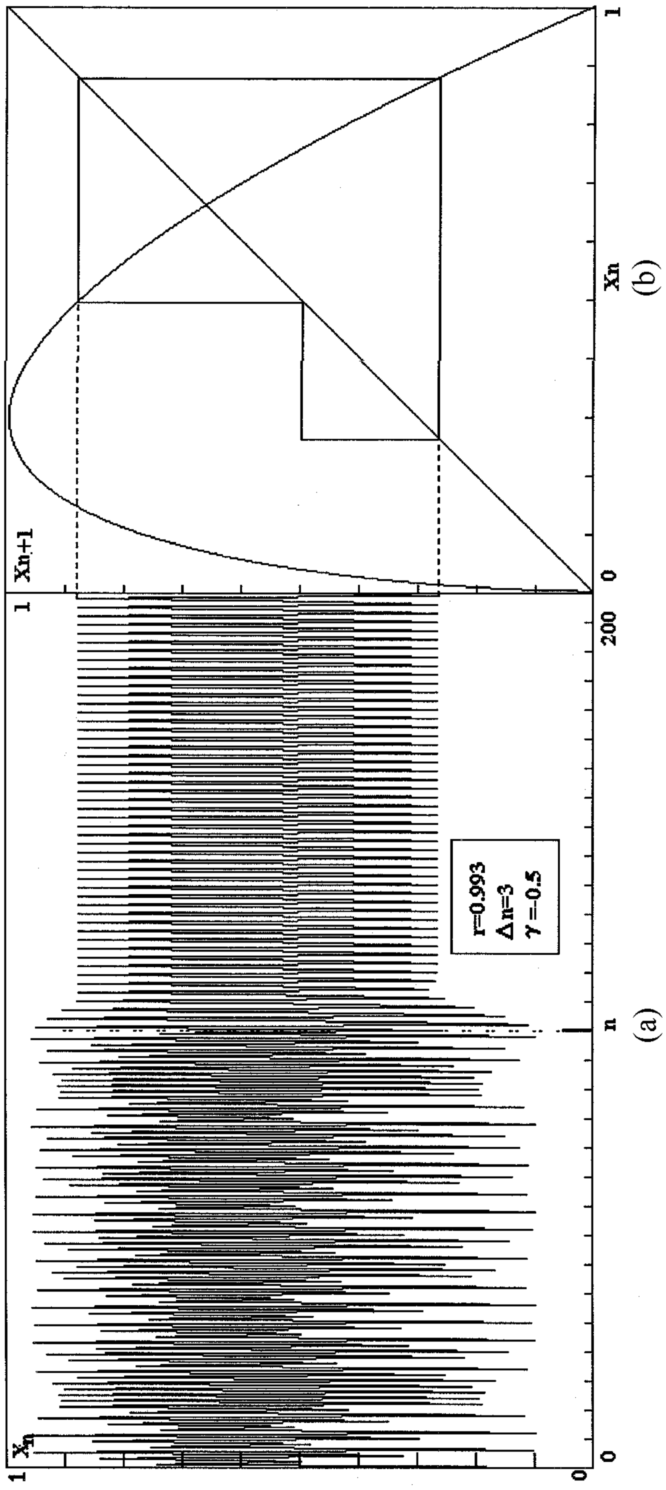

Figure 4. A stabilized one-period oscillation.

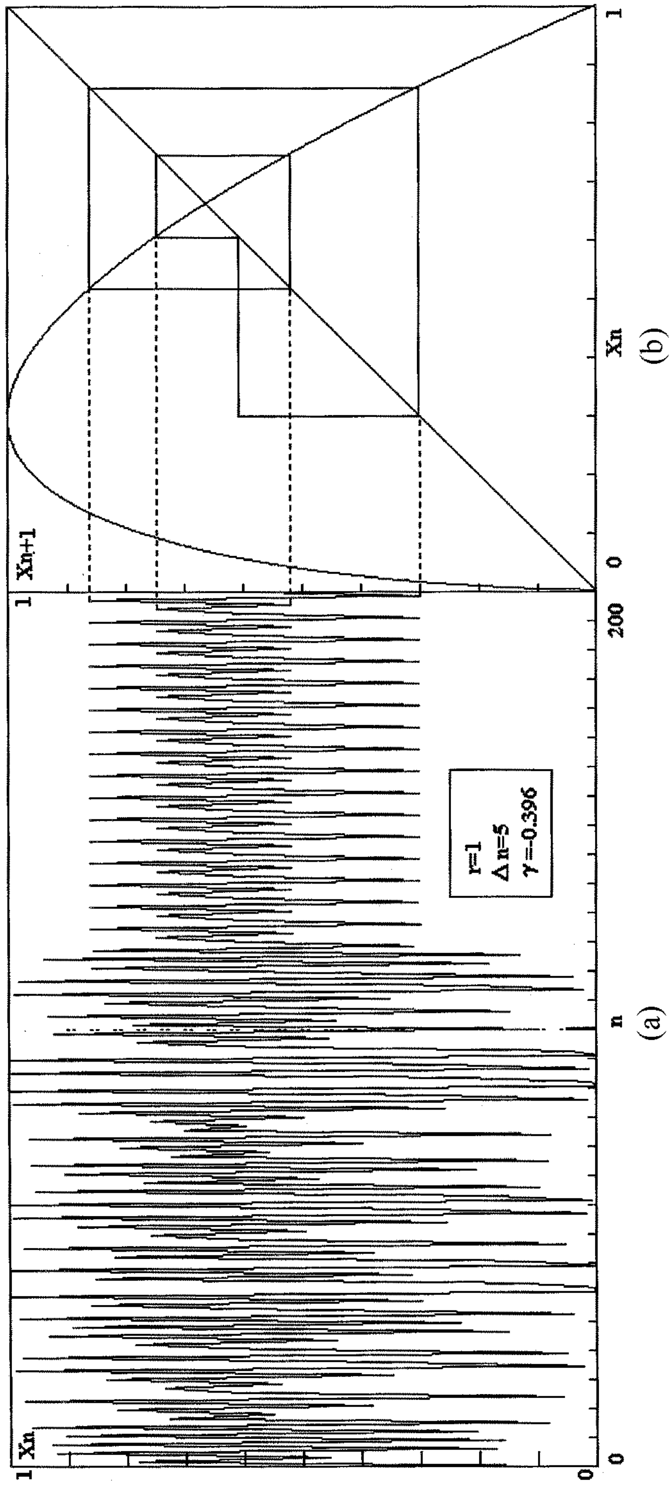

Figure 5. A stabilized two-periods oscillation.

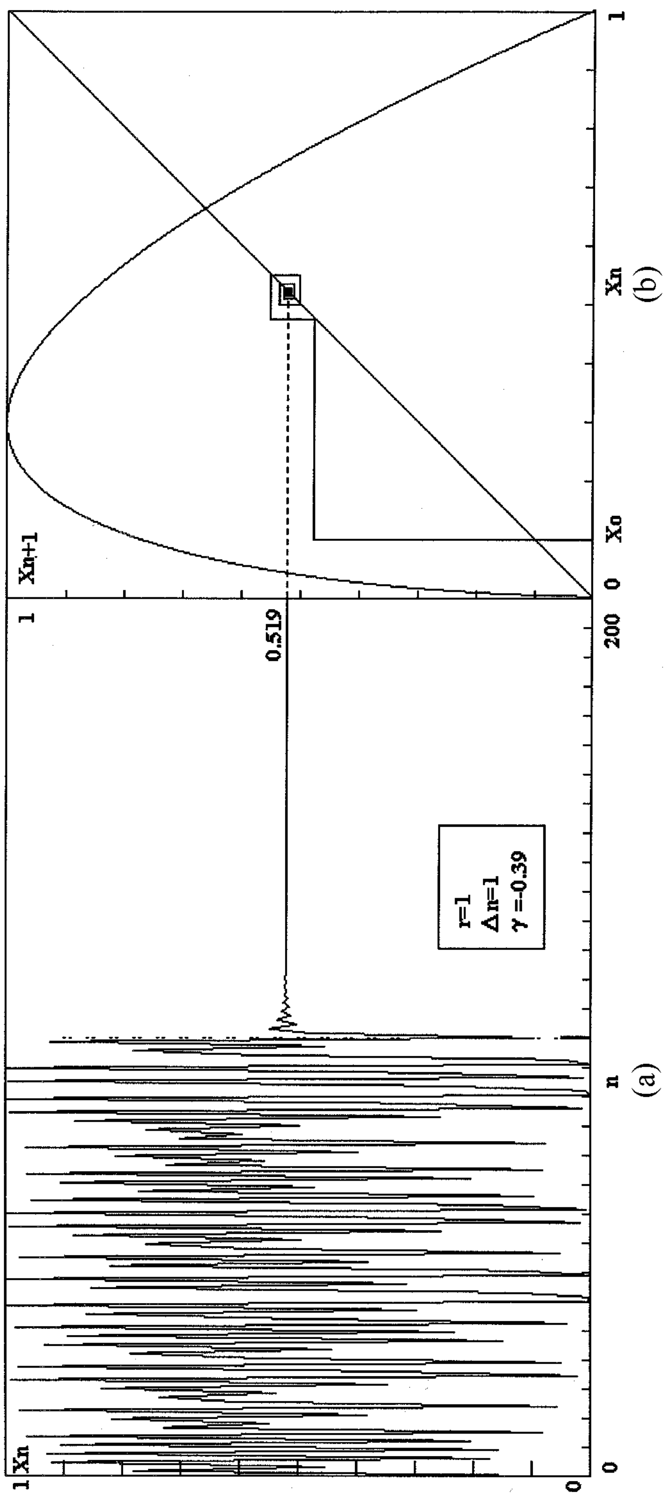

Figure 6. A stabilized steady state.

in plots of x_n versus n. Then after 100 iterations ($n = 100$), control is applied. The periodicity does not set in immediately. The vertical dashed lines indicate how many times, every Δn iterations, the feedback to the variable x is applied. The x_{n+1} versus x_n plots can be interpreted by observing that the cobweb method is applied to each new function created by the action of the control algorithm. At each moment of applied control, there is a jump in the variable. We note that, except in the case illustrated in Figure 6, the transients of 1000 iterations were left in the calculations, although the stabilisation of the periodic orbits can be seen after a smaller number of iterations.

5. Conclusions

In this study we present an application of controlling chaos in a one-dimensional model of malignant tumor growth. We chose the (GM) method, which consists in the application of a pulse of intensity γ every Δn iterations in the system variable and we were able to stabilize a steady state with different periodic motions embedded in the chaotic attractor. This method, which differs from the standard OGY one, is suitable for application to systems whose parameter values are held fixed, or when finding accessible system parameters may be a difficult task. In the particular case of malignant tumor growth, this control method could be thought as diminishing the concentration of cancerous cells by some amount. As a result, at least theoretically, there is the possibility of stabilising tumor growth to some periodic oscillation, or even to an equilibrium state. We conclude that the control of chaotic systems offers a new perspective of understanding biological systems, as nonlinear dynamical systems which react to a changing environment.

References

1. Ott, E., Grebogi, C. and Yorke, J.A.: *Phys. Rev. Lett.* **64** (1990), 1196.
2. Ditto, W.L., Rauseo, S.N. and Spano, M.L.: *Phys. Rev. Lett.* **65** (1990), 3211.
3. Singer, J., Wang, Y.Z. and Bau, H.H.: *Phys. Rev. Lett.* **66** (1991), 1123.
4. Azevedo, A. and Rezende, S.M.: *Phys. Rev. Lett.* **66** (1991), 1342.
5. Güémez, J. and Matías, M.A.: *Phys. Lett.* **A 181** (1993), 29.
6. Matías, M.A. and Güémez, J.: *Phys. Rev. Lett.* **72** (1994), 1455.
7. Güémez, J., Gutierrez, J.M. and Matías, M.A.: *Phys. Lett.* **A 190** (1994), 429.
8. Güémez, J., Gutierrez, J.M. and Matías, M.A.: *Physica* **D 79** (1994), 164.
9. Güémez, J. and Matías, M.A.: *Phys. Rev.* **E 52** (1995), R2145.
10. Matías, M.A. and Güémez, J.: *Phys. Lett.* **A 209** (1995), 48.
11. Weldon, T.E.: *Mathematical Models in Cancer Research*. Bristol: Adam Hilger, 1988.
12. Swan, G.W.: *Math. Biosciences* **101** (1990), 237.
13. Ahmed, E.: *Internat. J. of Theor. Phys.* **32** (1993), 353.

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